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THE STABILITY OF LINEAR POTENTIAL GYROSCOPIC SYSTEMS WHEN THE POTENTIAL ENERGY HAS A MAXIMUM†

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A brief review of the stability of linear systems acted upon by potential and gyroscopic forces is given. Some assertions on the stability of these systems are proved using Lyapunov's direct method. In particular, the necessary and sufficient conditions of stability for the systems investigated previously in [1, 2] are obtained. © 1997 Elsevier Science Ltd. All rights reserved.

1. The equations of the perturbed motion of a linear mechanical system acted upon by potential and gyroscopic forces can be reduced to the form

$$\ddot{q}+G\dot{q}+Kq=0 \tag{1.1}$$

Here q is an n-dimensional vector and $G^T = -G$, $K^T = K$ are constant matrices representing the gyroscopic and potential forces, respectively. We will assume that the matrix K is negative-definite. It follows from the Thomson–Tait–Chetayev result (see, for example, [3, 4]) that for odd n system (1.1) is unstable, while for even n it can be stable (i.e. gyroscopic stabilization is possible).

The following problem therefore arises, which is of practical importance: in the case of even n, it is required to determine the nature of the stability of system (1.1) with respect to the coefficients of the acting forces.

We will recall some of the few results obtained in this area.

1. If $4K - G^2 < 0$, the system will be unstable [5] (see also [6, 7]).

2. If $KG = GK$, the system is stable if and only if $4K - G^2 > 0$ [8].

Note that the condition for the matrices K and G to be commutative is extremely limiting. For example if $n = 2$, we have $K = \alpha I$, where α is a scalar and I is the identity matrix.

Using Lyapunov's function in the form of a quadratic integral of system (1.1) several results have been obtained [1, 9–11] that are intimately related to the proof of necessity and sufficiency, obtained by Lyapunov, of the condition for the existence of a positive-definite quadratic integral for the stability of system (1.1) (see also [12]).

When $G = \gamma G_0$, $\det G_0 \neq 0$, an estimate of the parameter γ, sufficient for gyroscopic stabilization to occur, was obtained in [9].

The following assertion gives a more accurate estimate.

3. If $\gamma^2 > 4k_+/g_-$, where $k_+(g_-)$ is the maximum (minimum) eigenvalue of the matrix $-K(G_0^2)$, the system is stable [10].

4. If $\alpha \in \mathbb{R}$ exists such that $K^{-1}(K-\alpha I) > 0$ and $K-\alpha I+\alpha G(K-\alpha I)^{-1}G > 0$ $(I-\alpha K) > 0$ and $4K(I-\alpha K)^{-1} - G(I-\alpha K)^{-1}G < 0$, the system is stable (unstable) [11].

Note that the last assertion on instability generalizes result 1 and is identical with it when $\alpha = 0$.

5. Suppose the matrix K is diagonal and suppose D is a certain diagonal positive-definite matrix. If D commutes with G and $K - D - G^2 + G(I-DK)^{-1}G > 0$, the system is stable [1].

It is important to note that the use of criteria 4 and 5 requires an investigation of the parameter α and the matrix D, respectively.

Fairly simple stability criteria were proposed [13, 2]. Their proof uses the following assertion: system (1.1) is stable when the roots of the characteristic equation are pure imaginary. Unlike a potential system, for system (1.1) this assertion is incorrect, as is indicated by the example of a system with two degrees of freedom

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$$\ddot{q}+2\begin{Vmatrix}0 & -1\\ 1 & 0\end{Vmatrix}\dot{q}-\begin{Vmatrix}1 & 0\\ 0 & 1\end{Vmatrix}q=0$$

In this example the quadratic elementary dividers correspond to the roots of the characteristic equation $\lambda_{1,2}=i$, $\lambda_{3,4}=-i$, and consequently the system is unstable.

In the following section we will use Lyapunov's direct method to obtain some results on the stability of system (1.1).

2. The following lemma, the proof of which can be found in [14], will be necessary later.

Lemma. For the quadratic form

$$\Phi=x^TAx+2x^TBy+y^TCy,\ \ A^T=A,\ \ C^T=C,\ \ x,y\in\mathbb{R}^n \tag{2.1}$$

to be positive-definite it is necessary and sufficient that the matrices A and $C-B^TA^{-1}B$ should be positive-definite. If $A>0$ and $C-B^TA^{-1}B\geqslant 0$, the form of (2.1) is positive semi-definite.

Theorem 1. If

$$2K-G^2-2k_+I>0 \tag{2.2}$$

where k_+ is the maximum eigenvalue of the matrix $-K$, system (1.1) is stable.

This theorem is analogous to the criterion proposed previously in [13]. We will compare them using the following example [13]

$$\ddot{q}+4\begin{Vmatrix}0 & -1\\ 1 & 0\end{Vmatrix}\dot{q}+\begin{Vmatrix}k_1-7 & 0\\ 0 & k_2-7\end{Vmatrix}q=0,\ \ k_i<7$$

It follows from Theorem 1, that this system is stable if $7>k_i>3$, while the criterion in [1] imposes the more rigid condition: $7>k_i>5$.

Proof of Theorem 1. System (1.1), as we know from [9], in addition to the energy integral $2H=\dot{q}^T\dot{q}+q^TKq$, also admits of the integral

$$\Gamma=(G\dot{q}+Kq)^T(G\dot{q}+Kq)+\dot{q}^TK\dot{q}$$

We will take as Lyapunov's function the bunch of these integrals $V=\Gamma-2k_+H$, which can be written in the form

$$V=\dot{q}\left(K-\frac{1}{2}G^2-k_+I\right)\dot{q}+F \tag{2.3}$$

$$F=-\frac{1}{2}\dot{q}^TG^2\dot{q}-2\dot{q}^TGKq+q^T(K^2-k_+K)q$$

Suppose condition (2.2) is satisfied. Since $G^2<0$ and $K^2+k_+K\leqslant 0$, by virtue of the lemma the quadratic form F is positive semi-definite and, consequently, since $(\dot{q}=0, F=0)=(0,0)$, the function (2.3) is positive-definite.

Hence, when condition (2.2) is satisfied the function V satisfied Lyapunov's theorem on stability, from which the assertion of Theorem 1 also follows.

Note 1. The condition of Theorem 1 is necessarily satisfied if criterion 3 holds.

Note 2. It can be shown that the conditions for the bunch of integrals $V=\Gamma-2\alpha H$, $\alpha\in\mathbf{R}$ to be positive-definite are equivalent to the conditions of stability of criterion 4, proved using a different form of Lyapunov's function.

Henceforth we will confine ourselves to considering classes of systems (1.1) for which

$$\det G\neq 0,\ KG^2=G^2K,\ KGKG^{-1}=G^{-1}KGK \tag{2.4}$$

For example, systems (1.1) with two degrees of freedom satisfy conditions (2.4).

System (1.1), (2.4) admit of the quadratic integral [15]

$$V = q^T(K+G^2-G^{-1}KG)Kq+4q^TKG\dot{q}+\dot{q}^T(K-G^2-GKG^{-1})\dot{q} \tag{2.5}$$

By virtue of the lemma, the form (2.5) is positive-definite if and only if the following two conditions are satisfied

$$LK>0 \tag{2.6}$$

$$M = L-2G^2+4GKL^{-1}G>0 \quad (L=L^T=K+G^2-GKG^{-1}) \tag{2.7}$$

We will analyse conditions (2.6) and (2.7). The first of these is equivalent to the condition $L < 0$, since $LK = KL$ and $K < 0$. Since $\det G \neq 0$, condition (2.7) is equivalent to the condition $G^TMG > 0$, which can be represented as

$$G^TMG = (L^2-4G^2K)L^{-1}G^2 = (L-2N)(L+2N)L^{-1}G^2>0 \tag{2.8}$$

where $N = (G^2K)^{1/2}$ is a positive-definite quadratic root of the positive-definite matrix G^2K. It is obvious that the conditions $L < 0$ and (2.8) can only be satisfied simultaneously if $L + 2N < 0$, i.e. the latter is the necessary and sufficient condition for integral (2.6) to be positive-definite. Consequently, by virtue of Lyapunov's theorem on stability we have the following theorem.

Theorem 2. If the matrix

$$K+G^2-GKG^{-1}+2(G^2K)^{1/2} \tag{2.9}$$

is negative-definite, system (1.1), (2.4) is stable.

Consider system (1.1) for which

$$G=\begin{Vmatrix} 0 & M \\ -M^T & 0 \end{Vmatrix}, \quad K=-\begin{Vmatrix} k_1I & 0 \\ 0 & k_2I \end{Vmatrix}; \quad k_1,k_2=\text{const}>0 \tag{2.10}$$

where I is the identity matrix and M is a non-degenerate $m \times m$ matrix, $2m = n$. Systems of this form were investigated previously in [2], where the condition $4K-G^2>0$ was proposed as the stability criterion.

It can be verified that the matrices (2.10) satisfy conditions (2.4) and, consequently, Theorem 2 is applicable. We can conclude from the fact that the matrix (2.9) is negative-definite, taking into account the structure of the matrices G and K, that

$$MM^T-2\sqrt{k_1}(MM^T)^{1/2}+(k_1-k_2)I>0$$

$$M^TM-2\sqrt{k_2}(M^TM)^{1/2}+(k_2-k_1)I>0$$

The last conditions can be reduced to the single condition

$$\Lambda \overset{\text{def}}{=} (MM^T)^{1/2}-\left(\sqrt{k_1}+\sqrt{k_2}\right)I>0 \tag{2.11}$$

Consequently, condition (2.11) is sufficient for system (1.1), (2.10) to be stable. It turns out that (2.11) is also the necessary condition for stability.

In fact, since MM^T is a symmetric positive-definite matrix, an orthogonal matrix U exists such that $U^TMM^TU = \operatorname{diag}(\mu_1^2,\ldots,\mu_m^2)$. Converting system (1.1), (2.10) by making the replacement

$$q=\begin{Vmatrix} U & 0 \\ 0 & U \end{Vmatrix}\begin{Vmatrix} x \\ y \end{Vmatrix}, \quad x,\ y\in\mathbf{R}^m$$

and eliminating y, we obtain

$$x^{(\mathrm{IV})}+(U^TMM^TU-(k_1+k_2)I)\ddot{x}+k_1k_2Ix=0 \tag{2.12}$$

We will assume that the matrix Λ is not positive-definite. Then, as an $i \in [1, \ldots, m]$ exists such that $|\mu_i| - (\sqrt{(k_1)} + \sqrt{(k_2)}) \leqslant 0$. With this condition, as can be shown, by investigating the corresponding equation of system (2.12)

$$x_i^{(IV)} + (\mu_i^2 - (k_1 + k_2))\ddot{x}_i + k_1 k_2 x_i = 0$$

the coordinate x_i is unstable.

Thus we have proved the following theorem.

Theorem 3. System (1.1), (2.10) is stable if and only if the matrix Λ is positive-definite.

We will now investigate the stability of system (1.1) when

$$G = \begin{Vmatrix} 0 & N \\ -N & 0 \end{Vmatrix}, \quad K = -\operatorname{diag}(k_1, \ldots, k_n) \tag{2.13}$$

$$N = \operatorname{diag}(\nu_i), \quad \nu_i \neq 0, \quad i = 1, \ldots, m, \quad k_j > 0, \quad j = 1, \ldots, n, \quad n = 2m$$

A single criterion of stability was established in [1] for this system. It turned out that the problem of the stability of system (1.1), (2.13) can be completely solved.

We will introduce the following notation: $|N| = \operatorname{diag}(|\nu_i|, \ldots, |\nu_m|)$, $D_1 = \operatorname{diag}(k_1, \ldots, k_m)$, $D_2 = \operatorname{diag}(k_{m+1}, \ldots, k_n)$. We can prove the following theorem in the same way as Theorem 3.

Theorem 4. System (1.1), (2.13) is stable if and only if the matrix $|N| - (D_1^{1/2} + D_2^{1/2})$ is positive-definite.

Note that the well-known condition for gyroscopic stabilization of system (1.1) with two degrees of freedom follows from Theorem 3 and also from Theorem 4 (see, for example, [3]): $|g| > \sqrt{(k_1)} + \sqrt{(k_2)}$, where g, k_1 and k_2 are the elements of the matrices G and $-K$. Hence, we can conclude that Theorems 3 and 4 are an extension of this condition to the case when $n > 2$.

In conclusion we will prove a criterion for the instability of system (1.1), (2.4) without additional assumptions regarding the structure of the matrices G and K.

Theorem 5. If

$$G^2 + 4(KGKG^{-1})^{1/2} > 0 \tag{2.14}$$

system (1.1), (2.4) is unstable.

Proof. Consider the alternating-sign function

$$V = 2q^T(KG - GK)q - q^T(G^2 + 4GKG^{-1})\dot{q} \tag{2.15}$$

The total derivative of function (2.15) with respect to time, by virtue of system (1.1), (2.4), has the form

$$\dot{V} = -\dot{q}^T(G + 4GKG^{-1})\dot{q} - \dot{q}^T G(4K + G^2)q + \dot{q}^T(G^2 + 4GKG^{-1})Kq \tag{2.16}$$

From conditions (2.4), taking into account the fact that the condition $G^2K = KG^2$ is equivalent to the condition $GKG^{-1} = G^{-1}KG$, we can conclude that the matrices $(G^2 + 4GKG^{-1})$ and $(G^2 + 4GKG^{-1})$ are symmetrical. Since $(G^2 + 4GKG^{-1} < 0)$, then, by virtue of the lemma, the form of (2.16) is positive-definite if and only if

$$16KGKG^{-1} - G^4 = (4(KGKG^{-1})^{1/2} - G^2)(4(KGKG^{-1})^{1/2} + G^2) > 0$$

Hence, when condition (2.14) is satisfied the function (2.16) satisfies the first Lyapunov theorem on instability, from which Theorem 5 also follows.

Note 3. It follows from the fact that the matrix is negative-definite that condition (2.14) is satisfied. When $KG \neq GK$ the inverse assertion does not hold.

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